Example Proof Segments of Proofs by (Regular) Mathematical Induction

Example Basis Steps:

Example Basis Step #1:

To Prove: For all integers n $\geqslant$ 1, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$.

Proof: (By Mathematical Induction)

[Basis Step] Let n = 1 .

Then, $1^3 + 2^3 + 3^3 + \cdots + n^3 = 1^3 = 1$

and $\left[\frac{n(n+1)}{2} \right]^2 = \left[\frac{1(1+1)}{2} \right]^2 = 1$.

$\therefore$ For n = 1 , $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$, by substitution.

[End of Basis Step]

Example Basis Step #2:

To Prove: $3^{2n} - 1$ is divisible by 8 , for each integer n $\geqslant$ 0 .

Proof: (By Mathematical Induction)

[Basis Step] Let n = 0 .

Then, $3^{2n} - 1 = 3^0 - 1 = 0$ and " 0 is divisible by 8 " .

$\therefore$ For n = 0 , " $3^{2n} - 1$ is divisible by 8 " , by substitution.

[End of Basis Step]

Example Inductive Step Openings

Example Inductive Step Opening #1:

To Prove: For all integers n $\geqslant$ 1, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$.

Proof: (By Mathematical Induction)

[Basis Step] . . . . [End of Basis Step]

[Inductive Step]

Let k be any integer such that k $\geqslant$ 1.

Suppose $1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2} \right]^2$. [Inductive Hypothesis]

[N T S : $\left(1^3 + 2^3 + 3^3 + \cdots + k^3\right) + \left(k + 1\right)^3 = \left[\frac{(k+1)\left((k+1) + 1\right)}{2} \right]^2$

[The Inductive Step continues from here.]

Example Inductive Step Opening #2:

To Prove: $3^{2n} - 1$ is divisible by 8 , for each integer n $\geqslant$ 0 .

Proof: (By Mathematical Induction)

[Basis Step]] . . . . [End of Basis Step]

[Inductive Step]

Let k be any integer such that k $\geqslant$ 0.

Suppose $3^{2k} - 1$ is divisible by 8 . [Inductive Hypothesis]

[N T S : $3^{2(k+1)} - 1$ is divisible by 8 .]

[The Inductive Step continues from here.]

Example Inductive Step Opening #3:

To Prove: $5^n + 9 < 6^n$, for all integers n $\geqslant$ 2.

Proof: (By Mathematical Induction)

[Basis Step]] ... ... [End of Basis Step]

[Inductive Step]

Let k be any integer such that k $\geqslant$ 2.

Suppose $5^k + 9 < 6^k$. [Inductive Hypothesis]

[N T S : $5^{(k+1)} + 9 < 6^{(k+1)}$.]

[The Inductive Step continues from here.]

Example Applications of the Inductive Hypothesis

Example Application of the Inductive Hypothesis #1:

To Prove: For all integers n $\geqslant$ 1, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2} \right]^2$.

Proof: (By Mathematical Induction)

. . .   .

[Inductive Step]

Let k be any integer such that k $\geqslant$ 1.

Suppose $1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\dfrac{k(k+1)}{2} \right]^2$. [Inductive Hypothesis]

[N T S : $\left(1^3 + 2^3 + 3^3 + \cdots + k^3\right) + \left(k + 1\right)^3 = \left[\dfrac{(k+1)\left((k+1) + 1\right)}{2} \right]^2$

[Now, if we add (k + 1)^3 to both sides of the equation in the Inductive Hypothesis, we obtain an equation which is closely related to the (N T S) equation that we need to show is true .]

[Continuing the proof ...]

By the Inductive Hypothesis, [adding (k + 1)^3 to both sides of its equation] ,

$$\therefore \left(1^3 + 2^3 + 3^3 + \cdots + k^3\right) + \left(k + 1\right)^3 = \left[\frac{k(k+1)}{2} \right]^2 + (k+1)^3 .$$

[The rest of the Inductive Step consists of algebraically proving that

$$\left[\frac{k(k+1)}{2} \right]^2 + (k+1)^3 = \left[\frac{(k+1)\left((k+1) + 1\right)}{2} \right]^2 ,$$

which is tedious but straightforward to accomplish.]

Example Application of the Inductive Hypothesis #2:

To Prove: $3^{2n} - 1$ is divisible by 8 , for each integer n $\geqslant$ 0 .

Proof: (By Mathematical Induction)

... ...

[Inductive Step]

Let k be any integer such that k $\geqslant$ 0.

Suppose $3^{2k} - 1$ is divisible by 8 . [Inductive Hypothesis]

[When working to prove a claim of divisibility by Mathematical Induction, immediately after making the Inductive Hypothesis, apply the definition of "divisibility (here by 8)" with the existence claim of the definition.]

$\therefore$ By the Inductive Hypothesis, there exists an integer t such that $3^{2k} - 1 = 8\text{t}$.

[In these "divisibility" proofs, you should immediately use the equation in the Inductive Hypothesis to solve for the term of highest degree in terms of the lower-degree terms.]

$\therefore 3^{2k} = 8\text{t} + 1$ by rules of algebra.

[N T S : $3^{2(k+1)} - 1 = 8\text{s}$ for some integer s . The proof could continue as follows:]

Since $3^{2(k+1)} - 1 = 3^{(2k+2)} - 1 = \left(3^2\right)\left(3^{2k}\right) - 1$

$= 9\left(3^{2k}\right) - 1 = 9\left(8t + 1\right) - 1$ by the Inductive Hypothesis,

$= 72t + 9 - 1 = 8(9t) + 8 = 8(9t + 1),$

$\therefore 3^{2(k+1)} - 1 = 8s, \quad where \ \ s = 9t + 1, \ an \ \text{int} eger .$

$\therefore 3^{2(k+1)} - 1$ is divisible by 8 .

Example Application of the Inductive Hypothesis #3:

To Prove: $5^n + 9 < 6^n$, for all integers n $\geqslant$ 2.

Proof: (By Mathematical Induction)

... ...

[Inductive Step]

Let k be any integer such that k $\geqslant$ 2.

Suppose $5^k + 9 < 6^k$. [Inductive Hypothesis]

[N T S: $5^{(k+1)} + 9 < 6^{(k+1)}$.]

[Quite often, when applying the Inductive Hypothesis to prove "comparison statements involving an inequality, we algebraically manipulate both sides of the (n = k) inequality in the Inductive Hypothesis (either multiplying both sides by the same number, or by adding the same term to both sides), and the goal in doing so is to make the left or right side (here, the left side) of the new inequality equal to the corresponding side of the (n = k + 1) inequality in the N T S statement. The other side of the new inequality is an intermediate value which can be shown to have the appropriate comparison (here "<") with the other side of the (n = k + 1) inequality in the N T S statement. The complete (n = k + 1) inequality in the N T S statement now follows by transitivity of inequality.]

[Continuing the proof ...]

By the Inductive Hypothesis, $5\left(5^k + 9\right) < 5\left(6^k\right)$ [from multiplying both sides by 5].

$\therefore\ 5 \cdot 5^k + 5 \cdot 9 < 5\left(6^k\right)$, and $\therefore\ 5 \cdot 5^k + 45 < 5\left(6^k\right)$.

Thus, $5^{(k+1)} + 9 + 36 < 5\left(6^k\right)$, and $\therefore\ 5^{(k+1)} + 9 < 5\left(6^k\right) - 36$.

Therefore, since also $5\left(6^k\right) - 36 < 5 \cdot 6^k < 6 \cdot 6^k = 6^{(k+1)}$,

$5^{(k+1)} + 9 < 6^{(k+1)}$ by transitivity of "less than"

[which is the (n = k+1) inequality in the N T S statement.] .

Example Closings of the Inductive Step and Proof Endings

Example Closing of the Inductive Step and Proof Ending #1 :

To Prove: For all integers n $\geqslant$ 1, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$.

Proof: (By Mathematical Induction)

. . .
. . .
. . .

[Inductive Step]

Let k be any integer such that k $\geqslant$ 1.

Suppose $1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2} \right]^2$. [Inductive Hypothesis]

. . .
. . .

$$\therefore \left(1^3 + 2^3 + 3^3 + \cdots + k^3\right) + \left(k + 1\right)^3 = \left[\frac{(k+1)\left((k+1) + 1\right)}{2} \right]^2 .$$

$\therefore$ For every integer k such that k $\geqslant$ 1 ,

if $1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2} \right]^2$,

then $\left(1^3 + 2^3 + 3^3 + \cdots + k^3\right) + \left(k + 1\right)^3 = \left[\frac{(k+1)\left((k+1) + 1\right)}{2} \right]^2$.

[End of the Inductive Step]

$$\therefore 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2 ,$$

for all integers n, by the Principle of Mathematical Induction. QED

Example Closing of the Inductive Step and Proof Ending #2 :

To Prove: $3^{2n} - 1$ is divisible by 8 , for each integer n $\geqslant$ 0 .

Proof: (By Mathematical Induction)

. . .
. . .
. . .

[Inductive Step]

Let k be any integer such that $k \geq o$.

Suppose $3^{2k} - 1$ *is divisible by* 8. [Inductive Hypothesis]

. . .
. . .
. . .

$\therefore$ $3^{2(k+1)} - 1$ *is divisible by* 8.

$\therefore$ For every integer k such that k $\geqslant$ 0, *if* $3^{2k} - 1$ *is divisible by* 8,

then $3^{2(k+1)} - 1$ *is divisible by* 8.

[End of the Inductive Step]

$\therefore$ $3^{2n} - 1$ *is divisible by* 8, *for each* int*eger* $n \geq 0$,

by the Pr*inciple of Mathematical Induction*. *QED*